Mathematics Questions

1. (a) $A = \begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 3 \\ 1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 16 & 4 \\ 4 & 1 \end{bmatrix}$

(i) $A - B = \begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix} - \begin{bmatrix} -3 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$

we subtract corresponding elements of the matrices

(ii) B^T, we flip matrix B over its diagonal, ie. switch the row and column indices of matrix B

Therefore $B^T = \begin{bmatrix} -3 & 1 \\ 3 & 4 \end{bmatrix}$

(iii) AB, we carry out multiplication a row by a column.

$$\begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} (3\times -3 + 3\times 1) & (3\times 3 + 3\times 4) \\ (1\times -3 + 8\times 1) & (1\times 3 + 8\times 4) \end{bmatrix} = \begin{bmatrix} -6 & 21 \\ 5 & 35 \end{bmatrix}$$

(iv) AC, row by column

$$\begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} (3\times 1 + 3\times 2) & (3\times 2 + 3\times 5) & (3\times 2 + 3\times 7) \\ (1\times 1 + 8\times 2) & (1\times 2 + 8\times 5) & (1\times 2 + 8\times 7) \end{bmatrix}$$

$$AC = \begin{bmatrix} 9 & 21 & 27 \\ 17 & 42 & 58 \end{bmatrix}$$

(v) $|A| = \det(A) = \begin{vmatrix} 3 & 3 \\ 1 & 8 \end{vmatrix}$, determined by taking the difference between the product of elements in the main diagonal - product of elements in the minor diagonal

$$\begin{vmatrix} 3 & 3 \\ 1 & 8 \end{vmatrix} = (3\times 8) - (1\times 3) = 24 - 3 = 21$$

(vi) $|C| = 0$, determinant is only sought for square matrix. C is a 2×3 which makes it have no determinant

(vii) A^{-1} is the inverse of matrix A.

First we get the determinant

$$|A| = \begin{vmatrix} 3 & 3 \\ 1 & 8 \end{vmatrix} = 21$$

$$A^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 8 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 8/21 & -1/7 \\ -1/21 & 1/7 \end{bmatrix}$$

(viii) $$D^{-1} = \frac{1}{|D|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow |D| = \begin{vmatrix} 16 & 4 \\ 4 & 1 \end{vmatrix} = 16 - 16 = 0$$

If the determinant is zero, the matrix has no inverse.

(ix) $$A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 8/21 & -1/7 \\ -1/21 & 1/7 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 8/21 & -1/7 \\ -1/21 & 1/7 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} (8/21 \times 3 + -1/7) & (8/21 \times 3 + -1/7 \times 8) \\ (-1/21 \times 3 + 1/7 \times 1) & (-1/21 \times 3 + 1/7 \times 8) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $$E = \begin{bmatrix} 5 & -2 & 0 \\ 3 & 1 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

$$|E| = +5(5-6) + 2(15-10) + 0(9-5)$$

$|E| = -5 + 10 = 5 \neq 0$, hence the rank of this matrix is 3

(C)(i) $q_1 = -3x^2 - y^2 + 6xz - 2z^2$ This is not quadratic polynomial, it cannot be factored

(ii) $q_2 = 3x^2 + 10yx + 3xy^2$; we divide through by x or factorize x

$q_2 = x(3x + 10yx + 3y^2)$, this cannot be factorised because the order of x is 1 instead of 2. Has positive definiteness

(ii) $q_3 = 4x^2 + 2y^2 + 8xy$

Rearranging : Factorising 2

$4x^2 + 8xy + 2y^2 = 2(2x^2 + 4xy + y^2)$ • This has positive indefiniteness.

2. (a)

(1) $10x_1 - 4x_2 + 6x_3 = 12$

$2x_1 + 3x_2 - 5x_3 = 0$

$4x_1 - 5x_2 + 6x_3 = 5$

$$A = \begin{bmatrix} 10 & -4 & 6 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 12 \\ 0 \\ 5 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B$

$$|A| = \begin{vmatrix} 10 & -4 & 6 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{vmatrix} = 10 \times \begin{vmatrix} 3 & -5 \\ -5 & 6 \end{vmatrix} + 4 \times \begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix} + 6 \times \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix}$$

$$|A| = 10(18 - 25) + 4(32) + 6(-22) = -74$$

$$\text{Adj}(A) = \text{Adj}\begin{bmatrix} 10 & -4 & 6 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 3 & -5 \\ -5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} \\ -\begin{vmatrix} -4 & 6 \\ -5 & 6 \end{vmatrix} & +\begin{vmatrix} 10 & 6 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 10 & -4 \\ 4 & -5 \end{vmatrix} \\ +\begin{vmatrix} -4 & 6 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 10 & 6 \\ 2 & -5 \end{vmatrix} & +\begin{vmatrix} 10 & -4 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T$$

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$$= \begin{bmatrix} -7 & -32 & -22 \\ -6 & 36 & 34 \\ 2 & 62 & 38 \end{bmatrix}^T = \begin{bmatrix} -7 & -6 & 2 \\ -32 & 36 & 62 \\ -22 & 34 & 38 \end{bmatrix}$$

To get the transpose, rows are rearranged in the column section.

Now $A^{-1} = \frac{1}{|A|} \times \text{adj}(A)$

Here $X = \frac{1}{|A|} \times \text{adj}(A) \times B$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{-74} \begin{bmatrix} -7 & -6 & 2 \\ -32 & 36 & 62 \\ -22 & 34 & 38 \end{bmatrix} \begin{bmatrix} 12 \\ 0 \\ 5 \end{bmatrix}$$

$$= -0.0135 \begin{bmatrix} (-7\times12 + -6\times0 + 2\times5) \\ (-32\times12 + 36\times0 + 62\times5) \\ (-22\times12) + 34\times0 + 38\times5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{-74} \begin{bmatrix} -74 \\ -74 \\ -74 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$

(ii) $0.3Y = 252 - 100r$

$0.25Y = 176 + 200r$

$0.3Y + 100r = 252$

$0.25Y - 200r = 176$

$$A = \begin{bmatrix} 0.3 & 100 \\ 0.25 & -200 \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} \qquad B = \begin{bmatrix} 252 \\ 176 \end{bmatrix}$$

$AX = B$

$X = A^{-1} B$

$$|A| = \begin{vmatrix} 0.3 & 100 \\ 0.25 & -200 \end{vmatrix} = 0.3 \times -200 - 100 \times 0.25 = -60 - 25 = -85$$

$$\text{Adj}(A) = \text{Adj} \begin{bmatrix} 0.3 & 100 \\ 0.25 & -200 \end{bmatrix} = \begin{bmatrix} -200 & -0.25 \\ -100 & 0.3 \end{bmatrix}^T = \begin{bmatrix} -200 & -100 \\ -0.25 & 0.3 \end{bmatrix}$$

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Now $A^{-1} = \frac{1}{|A|} \times adj(A)$

$$X = A^{-1} \cdot B$$

$$X = \frac{1}{|A|} \times adj(A) \cdot B$$

$$\begin{bmatrix} y \\ r \end{bmatrix} = \frac{1}{-85} \begin{bmatrix} -200 & -100 \\ -0.25 & 0.3 \end{bmatrix} \begin{bmatrix} 252 \\ 176 \end{bmatrix}$$

$$\begin{bmatrix} y \\ r \end{bmatrix} = -\frac{1}{85} \begin{bmatrix} -200 \times 252 + -100 \times 176 \\ -0.25 \times 252 + 0.3 \times 176 \end{bmatrix}$$

$$\begin{bmatrix} y \\ r \end{bmatrix} = -\frac{1}{85} \begin{bmatrix} -68000 \\ -10.2 \end{bmatrix} = \begin{bmatrix} 800 \\ 0.12 \end{bmatrix}$$

$y = 800, \quad r = 0.12.$

(b) $A = \begin{bmatrix} 0.3 & 100 \\ 0.125 & -200 \end{bmatrix} \quad B = \begin{bmatrix} 252 \\ 176 \end{bmatrix}$

The equations can be expressed as:

$$0.3y + 100r - 252 = 0$$
$$0.25y - 200r - 176 = 0$$

$$D_y = \begin{vmatrix} 100 & -252 \\ -200 & -176 \end{vmatrix} = 100 \times -176 - (252 \times 200)$$

$$D_y = -68000$$

$$D_r = \begin{vmatrix} 0.3 & -252 \\ 0.25 & -176 \end{vmatrix} = 0.3(-176) - 0.25(-252) = 10.2$$

$$D = \begin{vmatrix} 0.3 & 100 \\ 0.25 & -200 \end{vmatrix} = 0.3 \times (-200) - 0.25 \times 100 = -60 - 25 = -85$$

$$\frac{y}{D_y} = \frac{-r}{D_r} = \frac{1}{D} \quad \therefore \frac{y}{-68000} = \frac{-r}{10.2} = \frac{1}{-85}$$

$$y = \frac{-68000}{-85}, \quad r = \frac{-10.2}{-85}, \quad y = 800, \quad r = 0.12$$

3. (a) (i) $y = 2x^6 - 3x + \frac{1}{x^4} + 42$. $\qquad \frac{1}{x^4} = x^{-4}$

$$\frac{dy}{dx} = 12x^5 - 3 + -4x^{-5}$$

$$\frac{dy}{dx} = 12x^5 - \frac{4}{x^5} - 3$$

(ii) $4e^{x^2+7x} = 4\left(e^{x^2+7x}\right)$

$$\frac{dy}{dx} e^{f(x)} = f'(x)\, e^{f(x)}$$

$$\frac{dy}{dx} = 4(2x+7)\, e^{x^2+7x}$$

(iii) $y = \frac{4e^x - x^5}{x^3+2}$ $\qquad$ Using quotient rule $\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2}$

$$\frac{dy}{dx} = \frac{(x^3+2)\frac{d}{dx}\left(4e^x - x^5\right) - (4e^x - x^5)\left(\frac{d}{dx}(x^3+2)\right)}{(x^3+2)^2}$$

$$\frac{dy}{dx} = \frac{x^3+2\left(4e^x - 5x^4\right) - \left(4e^x - x^5\right)(3x^2)}{(x^3+2)^2}$$

(iv) $y = (x-3)^3 \ln(2x)$

Using product rule $u'v + v'u = \frac{d}{dx}(vu)$

$$\frac{d}{dx}((x-3)^3 \ln 2x + (x-3)^3 \frac{d}{dx}(\ln(2x))$$

$$3(x-3)^{3-1} \frac{d}{dx}(x-3) \ln(2x) + (x-3)^3 \frac{1}{2x} \frac{d}{dx}(2x)$$

$$3(x-3)^2 \ln(2x) + (x-3)^3 \cdot \frac{1}{2x} \cdot 2$$

$$\frac{dy}{dx} = 3(x-3)^2 \ln(2x) + \frac{(x-3)^3}{x}$$

(v) $y = \ln(6x^2 - 5x)$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln f(x)) = \frac{1}{f'(x)} \ln f(x) = \frac{1}{12x-5} \ln(6x^2-5x)$$

(b) (i) $y = -(x - 10x)^5$

$\frac{dy}{dx} = (-x + 10x)^5 = \frac{d}{dx}(9x)^5 = \frac{d}{dx}(59049x^5)$.

$\frac{d}{dx}(59049x^5) = 5 \times 59049x^4$

$= 295245x^4$

At turning point $dy/dx = 0$

$295245x^4 = 0$

$x = 0$

$\frac{d^2y}{dx^2} = 4 \times 295245x^3 = 1180980x^3 = 0$

Therefore this is an inflection point

Co-ordinate $= (0, 0)$

(ii) $y = -2x^2 + 3x + 10$

$\frac{dy}{dx} = -4x + 3$

$\frac{dy}{dx} = 0$ at turning point.

$-4x + 3 = 0$

$x = 3/4$, ~~being positive~~ $y = -2(3/4^2) + 3(3/4) + 10 = 11.125$

Co-ordinate $= (3/4, 11.125)$

$\frac{d^2y}{dx^2} = -4$ and therefore the co-ordinate above is a maximum point.

c) $R = 200Q - 3Q^2$

$C = 3Q^2 + 20Q + 40$

(i) $P(Q) = R(Q) - C(Q)$

$= (200Q - 3Q^2) - (3Q^2 + 20Q + 40)$

$= 180Q - 6Q^2 - 40$

(ii) $P(Q) = 180Q - 6Q^2 - 40$

$\frac{dP(Q)}{dQ} = 180 - 12Q$

At max profit $\frac{dP(Q)}{dQ} = 0$. $180 - 12Q = 0$.

$12Q = 180 \Rightarrow Q = 15$ Units.

Statistics

1. (a)

Student	1	2	3	4	5	6	7	8
Dev Econ	63	68	72	78	70	72	74	56

(i) Mean $= \frac{\text{Total}}{8} = \frac{1}{8}(63+68+72+78+70+72+74+56) = 69.125$

(ii) Median $k = \frac{1}{2}(n+1) = \frac{1}{2}(9) = 4.5$

data	56	63	68	70	72	72	74	78
Rank	1	2	3	4	5.5	5.5	7	8

Median $= 70 + (4.5-4)(72-70) = 70 + 0.5(2) = 71$

Mode $= 72$ (occurs twice)

(iii) Range $=$ Max $-$ Min $= 78 - 56 = 22$

+ Variance $= \frac{\Sigma(x-\bar{x})^2}{n}$

x	$x-\bar{x}$	$(x-\bar{x})^2$
63	-6.125	37.5156
68	-1.125	1.2656
72	2.875	8.2656
78	8.875	78.7656
70	0.875	0.7656
72	2.875	8.2656
74	4.875	23.7656
56	-13.125	172.2656

$\frac{\Sigma(x-\bar{x})^2}{n} = \frac{330.875}{8} = 41.35$

Variance $= \frac{330.875}{8} = 41.36$

Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{330.875}{8}} = 6.4311$

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Coefficient of variation $CV = \frac{s}{\bar{X}} \times 100 = \frac{6.4311}{69.125} \times 100\%$

$CV = 9.304$

(b) The mean has a higher strength because the data is distributed over a wider range of 22, followed by the median and the mode has the least strength.

c) Covariance is a measure of how two variables change together but its magnitude is unbounded. Correlation coefficient is the normalised version of the statistic. We get correlative coefficient by dividing covariance with the product of the two standard deviations.

Covariance tells us the direction one subject changes when the other one changes while correlation coefficient tells us the change in development economics leads by how much proportion change in the second variable.

2. (c) (i) standard deviation measures the amount of variability from the individual data values to the mean while standard error measures how far the sample mean is likely to be from the true population mean

(ii) If a population with mean μ and standard deviation σ and we take sufficiently large random samples from the population with replacement, then the distribution of the sample mean will be approximately normally distributed.

(d) (i) Sampling error - A statistical error that occurs when an analyst does not select a sample that represents the entire population of data

(ii) Bias of an estimator is the difference between an estimator's expected value and the true value of the parameter being estimated.

(iii) Mean squared error measures the average of the squares of the errors.

e) Type I error is the mistake of rejecting the null hypothesis when it is true and type II error is the mistake of accepting the null hypothesis when it is false.

Yes, If type I error increases, type II error decreases.

4 $n = 450$

$\mu = 35$

$s = 12$

(i) Confidence level ~~factor~~ $= 95\% = 0.95$

$\alpha = 0.05 \Rightarrow Z_{0.025} = \pm 1.96$

Confidence interval $(L_1, L_2) = \mu \pm Z_{0.05} \cdot \frac{s}{\sqrt{n}}$

$(L_1, L_2) = 35 \pm 1.96 \cdot \frac{12}{\sqrt{450}}$

$(L_1, L_2) = 35 \pm 1.1087 = 33.8913, 36.1087$

(ii) $P(X > 60)$

Z score $= \frac{60 - 35}{12/\sqrt{450}} = 44$

probability is zero since Z value is out of the distribution

b)(i) $\bar{x} = 35$ $n = 450$

$\mu = 40$ $s = 12$

Null hypothesis : $H_0 : \mu = 40$

$H_1 : \mu < 40$

(ii) $\alpha = 0.01$

$Z_{table} = -2.3264$

$$Z_{stat} = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{35-40}{12/\sqrt{450}} = -187.5$$

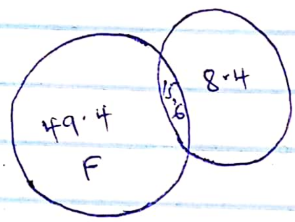
We therefore reject the null hypothesis and conclude that the average travelling time is less than 40 mins

5. 65% - At least one football match

24% - At least one cricket match

15.6% - At least one football match and one cricket match.

(a)



$n(F) = 65$

$n(C) = 24$

$n(F \cap C) = 15.6$

a) $P'(F) = 1 - P(F) = 1 - 0.65 = 0.35$

b) $n(C) = 24 - 15.6 = 8.4\% = 0.084$

(c) $P = 49.4 + 8.4 = 57.8\% = 0.578$

$P = P(F) + P(C) = 0.578$

(d) $P(F \cap C) = 15.6\% = 0.156$

(e) Events are independent if $P(A \cap B) = P(A)\,P(B)$

$0.156 \neq 0.494 \cdot 0.084$, Therefore the events are not independent.